DOCUMENT RESUME

ED 187 589 SE 031 108

TITLE The Arithmetic Project Course for Teachers - 8. Topic: Lower Brackets and Upper Brackets. Supplement: Arithmetic With Frames.

INSTITUTION Education Development Center, Inc., Newton, Mass.: Illinois Univ., Urbana.

SPONS AGENCY Carnegie Corp. of New York, N.Y.; National Science Foundation, Washington, D.C.

PUB DATE 73

NOTE 31p.; For related documents, see SE 031 100-120. Contains light and broken type.

EDRS PRICE MF01/PC02 Plus Postage.

DESCRIPTORS *Algorithms; Elementary Education; *Elementary School Mathematics; *Elementary School Teachers; Films; Grade 5; Inequality (Mathematics); Inservice Education; *Inservice Teacher Education; Mathematics Curriculum; *Mathematics Instruction; Mathematics Teachers; Number Concepts; *Problem Sets; Teacher Education

IDENTIFIERS *University of Illinois Arithmetic Project

ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises on lower brackets and upper brackets, a summary of problems from the film "Inequalities With Lower Brackets," and the supplement. (MK)

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THE ARITHMETIC PROJECT COURSE FOR TEACHERS

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TOPIC: Lower Brackets and Upper Brackets.

FILM: Inequalities With Lower Brackets, Grade 5

SUPPLEMENT: Arithmetic With Frames

8

NAME: .....

SE 031 108

This booklet is part of a course for teachers produced by The Arithmetic Project in association with Education Development Center. Principal financial support has come from the Carnegie Corporation of New York, the University of Illinois, and the National Science Foundation.

The course is available from:

THE ARITHMETIC PROJECT
Education Development Center
55 Chapel Street
Newton, Massachusetts 02160

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BOOK EIGHT

I. COMPUTING WITH LOWER BRACKETS

When working problems like these one should first look them over to determine how tricky they may be, and where the tricks are likely to occur. Are detailed computations going to be necessary? What do previous exercises contribute to the solution?

1. $\left\lfloor \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{108}{201} \right\rfloor =$ _____

2. $\left\lfloor \frac{1}{2} + \frac{1}{2} + \frac{101}{204} + \frac{1}{2} \right\rfloor =$ 1

3. $\left\lfloor \frac{1}{2} + \frac{1}{2} + \frac{100}{204} + \frac{1}{2} \right\rfloor =$ _____

4. $\left\lfloor \frac{1}{2} + \frac{1}{2} + \frac{102}{204} + \frac{1}{2} \right\rfloor =$ _____

5. $\left\lfloor \frac{1}{2} + \frac{1}{2} + \frac{310}{204} + \frac{1}{2} \right\rfloor =$ _____

6. $\left\lfloor \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{11}{30} \right\rfloor =$ _____

7. $\left\lfloor \left\lfloor 5\frac{2}{3} \right\rfloor + 5\frac{1}{3} + \left\lfloor 5\frac{2}{3} \right\rfloor + 5\frac{1}{2} \right\rfloor =$ _____

8. $\left\lfloor \left\lfloor 5\frac{1}{3} + 5\frac{1}{2} \right\rfloor + \left\lfloor 5\frac{2}{3} + 5\frac{1}{2} \right\rfloor + \frac{9}{10} \right\rfloor =$ _____

In each of the next four problems, the larger number should be encircled. The first one has been done for you.

9. $\left(\frac{1}{11}\right)$, $\frac{1}{12}$

10. $\frac{1}{400}$, $\frac{1}{401}$

11. $\frac{1}{20 + \frac{1}{2}}$, $\frac{1}{20 - \frac{1}{2}}$

12. $\frac{1}{1000 - \frac{1}{10}}$, $\frac{1}{1000}$

13. $\left[\frac{\left\lfloor 2 \times 6\frac{1}{7} \right\rfloor}{\left\lfloor 1\frac{1}{2} + 1\frac{3}{4} \right\rfloor} \right] =$ _____

14. $\left[\frac{\left\lfloor 2 \times 6\frac{1}{7} \right\rfloor}{\left\lfloor 1\frac{1}{2} + 1\frac{3}{4} \right\rfloor - \frac{1}{100}} \right] =$ _____

15. $\left[\frac{\left\lfloor 2 \times 6\frac{1}{7} \right\rfloor}{\left\lfloor 1\frac{1}{2} + 1\frac{3}{4} \right\rfloor + \frac{1}{100}} \right] =$ _____

II. UPPER BRACKETS

Here are some examples with upper brackets:

$$\lceil 7\tfrac{1}{2} \rceil = 8$$

$$\lceil 10 \rceil = 10$$

$$\lceil 13\tfrac{1}{18} \rceil = 14$$

Some students may feel that upper brackets should do something fancier than they do. In particular, some students may feel that $\lceil 17 \rceil$ should not be 17. (They may think it should be 18.) These students have probably decided without realizing it that $\lfloor \ \rfloor$ and $\lceil \ \rceil$ should always give different answers. But lots of different functions agree here and there; $\lfloor \ \rfloor$ and $\lceil \ \rceil$ agree on whole numbers.

Now do these:

1. $\lceil 100.2 \rceil =$

2. $\lceil 0 \rceil =$

3. $\lceil \frac{47}{6} \rceil =$

4. $\lceil \frac{46}{6} \rceil =$

5. $\lceil \frac{43}{6} \rceil =$

6. $\lceil \frac{42}{6} \rceil =$

7. $\left\lceil \frac{40}{6} \right\rceil$ =

8. $\left\lceil \frac{35}{6} \right\rceil$ =

9. $\left\lceil \frac{6,660}{6} \right\rceil$ =

10. $\left\lceil 7 + 2\frac{3}{8} \right\rceil + \square = 21$

11. $7 + \left\lceil 2\frac{3}{8} \right\rceil + \square = 21$

12. $9 + \left\lceil \frac{3}{8} \right\rceil + \square = 21$

13. $\left\lceil \square + \frac{9}{10} \right\rceil = \square + \left\lceil \frac{9}{10} \right\rceil$

Describe all the numbers that work:

14. $\left\lceil 3\frac{1}{4} \right\rceil - \left\lfloor 3\frac{1}{4} \right\rfloor$ =

15. $\left\lceil 501,630\frac{1}{12} \right\rceil - \left\lfloor 501,630\frac{1}{12} \right\rfloor$ =

16. $\left\lceil 12\frac{9}{10} \right\rceil - \left\lfloor 12\frac{19}{46} \right\rfloor$ =

17. $\left\lceil 4\frac{4}{7} \right\rceil \times 13 - \left\lfloor 4\frac{1}{7} \right\rfloor \times 13 =$

18. $178 \times \left\lceil \frac{58}{17} \right\rceil - 178 \times \left\lfloor \frac{58}{17} \right\rfloor =$

19. $\left\lceil 2 \times \square \right\rceil - \left\lfloor 2 \times \square \right\rfloor = 1$

Describe, somehow, all the numbers that work:

20. $\left\lceil 2 \times \square \right\rceil - \left\lfloor 2 \times \square \right\rfloor = 0$

What numbers work?

21. $\left\lceil 2 \times \square \right\rceil - \left\lfloor 2 \times \square \right\rfloor = 2$

What numbers work?

22. $5 \times \left\lceil 10 \right\rceil - 2 \times \left\lfloor 10 \right\rfloor =$

23. $3 \times \left\lceil 48\frac{1}{11} \right\rceil - 2 \times \left\lfloor 48\frac{1}{11} \right\rfloor =$

24. $2 \times \left\lceil 18\frac{7}{8} \right\rceil - \left\lfloor 18\frac{7}{8} \right\rfloor =$

25. $9 \times \left\lceil 4\frac{1}{3} \right\rceil - 8 \times \left\lfloor 4\frac{1}{3} \right\rfloor =$

26. $8 \times \left\lceil 4\frac{1}{3} \right\rceil - 5 \times \left\lfloor 4\frac{1}{3} \right\rfloor =$

27. Note that the answer to the problem above might be found by looking at it written as a big subtraction problem:

$$\left(\left\lceil 4\tfrac{1}{3} \right\rceil + \left\lceil 4\tfrac{1}{3} \right\rceil + \left\lceil 4\tfrac{1}{3} \right\rceil + \left\lceil 4\tfrac{1}{3} \right\rceil + \left\lceil 4\tfrac{1}{3} \right\rceil + \left\lceil 4\tfrac{1}{3} \right\rceil + \left\lceil 4\tfrac{1}{3} \right\rceil + \left\lceil 4\tfrac{1}{3} \right\rceil \right)$$

$$- \left(0 + 0 + 0 + \left\lfloor 4\tfrac{1}{3} \right\rfloor + \left\lfloor 4\tfrac{1}{3} \right\rfloor + \left\lfloor 4\tfrac{1}{3} \right\rfloor + \left\lfloor 4\tfrac{1}{3} \right\rfloor + \left\lfloor 4\tfrac{1}{3} \right\rfloor \right)$$

If you can, use some such method, rather than computing out in full detail, to do this:

$$117 \times \left\lceil 199\frac{3}{7} \right\rceil - 101 \times \left\lfloor 199\frac{3}{7} \right\rfloor =$$

28. Whatever grade you are now teaching, assume for the moment that you have recently introduced lower brackets to a fifth grade class. You would like the tenth in a sequence of problems to be:

$$2 \times \left\lfloor 5\frac{3}{4} + \frac{8}{9} + \frac{16}{17} \right\rfloor =$$

Write the nine problems that precede it (and give their answers).

Epilogue

By now you probably have noticed that many of the problems with upper or lower brackets induce the student to estimate with numbers in order to get an exact answer to a problem.

Consider this example:

$$\left\lfloor\ 2\frac{99}{100} + 10\frac{2}{3} + 10\frac{4}{5} + 10\frac{5}{6} + 10\frac{3}{4} + \frac{70}{100}\ \right\rfloor$$

Each fraction above is rather close to one but less than one. The student might follow a hunch and decide that the answer is 47, which is one short of

$$3 + 11 + 11 + 11 + 11 + 1\ .$$

But he should look again (not compute in full!). There are six numbers added. Several of the fractions involved are a good bit less than one. What can he say about

$$\frac{1}{100} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{4} + \frac{30}{100}\ ,$$

which is the amount by which the number inside the brackets misses 48? In particular, is it more than 1? More than 2?

One way to think about this is to look for thirds, or for sums approximating $\frac{1}{3}$:

$$\frac{1}{100} + \underbrace{\frac{1}{3}}_{\frac{1}{3}} + \underbrace{\frac{1}{5} + \frac{1}{6}}_{\text{A bit over } \frac{1}{3} \text{ (because } \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.)} + \underbrace{\frac{1}{4} + \frac{30}{100}}_{\text{Well over } \frac{1}{3}\text{. In fact, a bit over } \frac{1}{2}\text{, because } \frac{1}{4} = \frac{25}{100}.}$$

So we have more than 3 one-thirds or more than 1. Is it as much as 2? No, the $\frac{1}{100}$ and the other excesses over $\frac{1}{3}$ don't even get close to another 1. So,

$$\left\lfloor 2\frac{99}{100} + 10\frac{2}{3} + 10\frac{4}{5} + 10\frac{5}{6} + 10\frac{3}{4} + \frac{70}{100} \right\rfloor = 46$$

because the sum inside the brackets is down more than 1 but not as much as 2 from 48. After applying lower brackets the exact answer is 46.

This kind of estimation, reasoning, number sense, and rapid recognition of the relative sizes of fractions are some of the reasons for presenting upper and lower bracket problems to students.

* * *

Lower brackets is a standard mathematical function which is used in advanced mathematics and science. Its formal name is the "greatest integer not greater than" function. If you want to spend the time deciphering this phrase, you will see that it makes sense. But we suggest that you do not mention this tongue-twister to students.

If you have had contact with this function outside this course, you should know that it normally appears written as $\left[\ \right]$. The thick vertical bars distinguish these brackets from those used merely for grouping. We have used $\lfloor\ \rfloor$ because this symbol can be written easily on paper or on the chalkboard, and because it paves the way for the analogous upper brackets, $\lceil\ \rceil$. (Standard mathematical papers do not use a symbol synonymous with upper brackets because expressions involving upper brackets can be written using lower brackets and some cunning.) To make the ideas behind $\lfloor\ \rfloor$ and $\lceil\ \rceil$ clearer to

teachers and their students we have decided to use both kinds of symbols, but, again, these details are probably better not brought up with elementary school students.

Some of the Project's films and written materials refer to "square brackets", [ ]. These materials were prepared before we shifted to lower and upper brackets. Square brackets, [ ], are synonymous with lower brackets, ⌊ ⌋.

Summary of Problems in the Film "Inequalities With Lower Brackets"

5th Grade, Browne School, Watertown, Massachusetts
Teacher: Francis X. Corcoran

$8\frac{1}{3} + 7\frac{1}{3} =$

$48\frac{2}{3} + 7\frac{2}{3} =$

$23\frac{4}{5} + 23\frac{4}{5}$ ($48\frac{8}{5}$ is given.)

I see where you get the $\frac{8}{5}$ but not the 48. ($46\frac{8}{5}$)

("The $\frac{8}{5}$ can be reduced to $1\frac{3}{5}$. So it's $47\frac{3}{5}$.")

Is $46\frac{8}{5}$ a wrong answer?

$\square + \square = 5$ ($2\frac{1}{2}$)

$\square + \square = 5\frac{1}{2}$ ($2\frac{3}{4}$)

Use the answer that he gave and tell me another number you could have at the end besides $5\frac{1}{2}$. Do the same thing as he did when he got $46\frac{8}{5}$.

($4\frac{6}{4}$)

$\square + \square = 5\frac{1}{3}$ (Wrong answer of $2\frac{1}{8}$ is given.)

$(5\frac{2}{3} + 5\frac{2}{3}) \times 2 =$ ($22\frac{2}{3}$)

$(5\frac{2}{3} + 5\frac{2}{3}) \times 2 > \square$

A number that will work if I put it into the $\square$.

(20)

$(5\frac{2}{3} + 5\frac{2}{3}) \times 2 > \square + 20$ (2)

$(5\frac{2}{3} + 5\frac{2}{3}) \times 2 > \square + 24$ (-2)

Do the same problem (above), but now you can't use any negative numbers.

(0 is given.)

What does this say? ("$22\frac{2}{3}$ is bigger than 24."
"It's impossible.")

Why? ("If you can't use a negative number, $22\frac{2}{3}$ is less than 24 so it won't come out.")

$$\left\lfloor 99\tfrac{1}{5} + 98\tfrac{1}{5} + 101\tfrac{1}{5} + 102\tfrac{1}{5} + 100\tfrac{1}{6} \right\rfloor =$$

(Answers of 500, $499\frac{1}{5}$, 400, and 501 are given.)

("It has to be 500.")

Who can convince me? ("99 + 101 is 200, and 98 + 102 is 400, and 100 more is 500. $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5}$ and $\frac{1}{6}$ is smaller than $\frac{1}{5}$ so it can't be 501.")

$$\left\lfloor 99\tfrac{1}{5} + 98\tfrac{1}{5} + 101\tfrac{1}{5} + 102\tfrac{1}{5} + 100\tfrac{1}{\square} \right\rfloor = 501$$

($\frac{1}{5}$)*

Give me another number that I could put in the box besides 5.

($\frac{1}{10}$)

Why doesn't $\frac{1}{10}$ work? ("10 is bigger than 5, but $\frac{1}{10}$ is a smaller fraction.")

Can you give me a fraction bigger than $\frac{1}{5}$? ($\frac{1}{4}$)

---

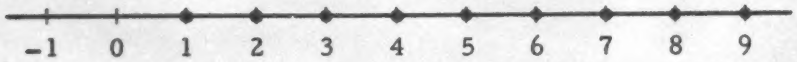
*Actually it is 5 that goes in □ and not $\frac{1}{5}$. Even though the problem asks for the denominator, answers which gave the entire fraction (with 1 as numerator) were accepted and discussed.

Yesterday you were doing this problem:

$$\left\lfloor \square + \frac{1}{2} \right\rfloor > \square$$

Does anyone remember the answer to this?

A student goes up to the board and puts this answer on the number line:



Another student objects: "If you put 3 in the box, 3 plus $\frac{1}{2}$ with those brackets behind it you take away the $\frac{1}{2}$ and then it says that 3 is bigger than 3. All the whole numbers don't work."

(The student at the board agrees that he is wrong and begins to fix up the answer.)

Who can tell which numbers work? ("All the halves will work.")

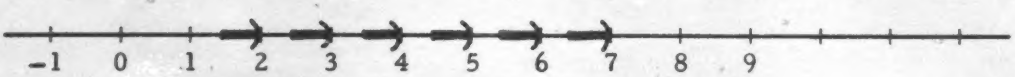
Is that all that will work? ($\frac{2}{3}$'s will work.)

Are those the only two places that will work?

(All the $\frac{3}{4}$'s will work.)

One sentence that will tell me everything that will work.

(Every number from the half mark to the whole mark, but not the whole numbers.)

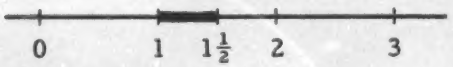


New problem: $\square > \left\lfloor \square + \frac{1}{2} \right\rfloor$

Who can tell me what numbers will work?

("From the whole number up to the half.")

Teacher draws a piece of the number line:



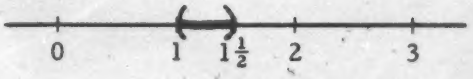
Remember, when we didn't want to count the 1, we put a little parenthesis mark before it. Do we need any of these here? Where do you have to put it?

("Just before the half.")

Is that all?

("Maybe before the whole number.")

Teacher draws in the parentheses:



Is that right?

Supplement

ARITHMETIC WITH FRAMES
(revised)

by

UICSM Project Staff*
University of Illinois, Urbana, Illinois

We offer these suggestions not as a comprehensive new program nor as answers to any specific problems, but as informal notes for those teachers who are interested in trying some new ideas. In the remainder of the article, for the sake of economy, we have omitted all tempering words; each declarative sentence is prefaced tacitly with some such expressions as "It seems to us that..." or "Our guess is...". Also we recognize that some teachers may respond to our recommendations with: "Why, I've been doing that for a long time." To such teachers: Greetings!

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* This supplement was originally written by the staff of the University of Illinois Committee on School Mathematics in 1957; at that time David Page was the editor of the UICSM Project. The supplement was substantially revised later by Prof. Page and the staff of the University of Illinois Arithmetic Project to accord with the experience of the Arithmetic Project. In its original form it appeared first as an article in The Arithmetic Teacher, Volume IV, No. 3, April 1957.

* * *

At each stage of their learning of arithmetic, children should be taught to do some corresponding "algebra". In this way, algebra grows naturally out of arithmetic and would not be viewed in high school as an entirely new subject in which, too often, the student thinks he "adds and subtracts letters". If a student knows that

$$5 + 3 = 8$$

then he is ready to be asked to use a number to replace the question mark in each of the following three expressions to make them into true statements:

$$? + 3 = 8$$

$$5 + ? = 8$$

$$5 + 3 = ?$$

In contrast the student should also be asked to make other replacements for each question mark to make false statements. In an attempt to teach students correct arithmetic facts, many teachers feel that a statement such as

$$5 + 6 = 8$$

should be avoided. Rather than avoiding it, a student should examine such a statement and simply declare that it is false (or "not-true", as some children prefer to say). When the student gets to high school mathematics, he will no longer be able to live in a utopia where false statements are never allowed to occur. They are necessary preliminaries to significant true state-

ments. Even in early grades a student should be encouraged to make estimates and enlightened guesses and then to check them. He must be prepared to find that he has a false statement. This tells him that his guess was not good enough and often gives a hint on how to correct it.

* * *

Elementary school students become too accustomed to the pattern:

$$\begin{array}{r} 2 \\ +\ 3 \\ \hline 5 \end{array}$$

From the beginning they should work with the pattern:

$$2 + 3 = 5$$

and should also work with the pattern:

$$5 = 2 + 3$$

We have asked many second and third graders if the following statement is true:

$$2 + 2 + 2 = 2 + 8$$

A surprising fraction of them, seeing four 2's and an 8 at the end, read it as "2 + 2 + 2 + 2 = 8" and say it is true. Students should learn the symmetry of the equal sign from the outset.

* * *

Rather than using question marks to give incomplete expressions such as $3 + ? = 5$, we propose the use of <u>frames</u>. Thus, students might be told to complete the following incomplete statements, first so that each of them becomes true, and later so that each of them becomes false:

$$\begin{array}{r} 3 \\ +\ \square \\ \hline 10 \end{array} \qquad\qquad 4 + \triangle = 9$$

$$\diamond + 6 = 13 \qquad\qquad 10 = 2 + \hexagon$$

After students have carried out these exercises, ask them to tell for a given exercise how many choices for replacement there are which will give true statements. If you want to introduce a bit of philosophical discussion in the arithmetic class, ask them to support the answer that in each one of the four cases above there is only one number which produces a true statement. [A thoughtful student may say that either 5 or 2 + 3 can go in the box to give a true statement. In this case he is giving two names for one number, just as William and Bill may be two names for one boy. But it is still one number as it is one boy.] Also, ask them how many choices they have for replacements which give false statements. Accept answers such as "many" or better yet "as many as you please" or "so many you couldn't even write them all down ever".

At a later stage, students can be introduced to equations which contain more than one frame. For example:

$$\square + \square = 12$$

When more than one frame occurs, the student must learn an important rule of the game. In the two boxes above, he must use the same number to fill each box. Thus, if he writes 3 in one of the boxes, to follow the rule he must write 3 (or 2 + 1, etc.) in the other box; he is not to write 3 in one of the boxes and 4 in the other box. This rule of "like" replacements holds only for frames of the same shape. Again, ask students to consider how many choices they have for making $\square + \square = 12$ into a true statement, and how many choices they have for making it false.

When students have had some experience in playing the "replacement game", you may find it convenient to use the word satisfy in such cases as:

the number 6 satisfies $\square + \square = 12$

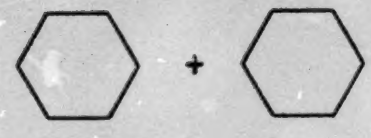
and:

7 does not satisfy $\triangle + 3 = 11$.

Now you can economize on words by saying that only one number satisfies $\square + \square = 12$ although any of that number's names could be used.

At various stages of his work in arithmetic, it would be appropriate for a student to be confronted with problems such as the following:

Find all the numbers which satisfy

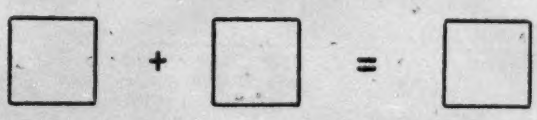
 = 16 ans. 8

= 15 ans. $7\frac{1}{2}$

1 = ans. $\frac{1}{3}$

16 = ans. 4 (If the student knows about negative numbers, also -4)

= 8 ans. 2

 ans. 0

Such a list can be extended indefinitely and as much difficulty as is appropriate can be introduced using fractions, decimals, and "big numbers". Mixed in with such exercises should be some equations such as:

5 + = 2 + + 3

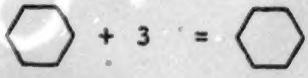
which are satisfied by every number. The student has solved the problem when he points out that every number satisfies it (or equivalently that no number fails to satisfy it). The fact that an expression is satisfied by every

number can be disguised as intricately as you please by increasing the complexity of the expressions. For example:

$$16.2 + \square + 8.3 + \square = \square + 36.9 + \square - 12.4$$

is satisfied by every number.

The student should also be given equations such as:



There is no number which satisfies this equation. Every number will make if false.

Another interesting equation is:

(Now, in substituting, the number to be tried must be used in all three diamonds.) Here, the student should observe that the numbers 0 and 1 satisfy the expression and that no other numbers satisfy it. Encourage students to argue in support of such a conclusion. Hope for statements such as, "Well, if you multiply any number bigger than 1 by itself, you get an even bigger number. If you multiply a number between 0 and 1 by itself, you get a smaller number." Such conclusions can be promoted by questions from the teacher such as:

> If you multiply 8 by a number larger than 1, what do you know about the answer?

If you multiply 1,632 by a number less than 1, what do you know about the answer?

In fact, as soon as a child can tell <u>without computation</u> which of the numbers $\frac{981}{972}$ and $\frac{981}{972} \times \frac{981}{972}$ is the larger, and whether $\frac{2355}{2356}$ is larger or smaller than $\frac{2355}{2356} \times \frac{2355}{2356}$, you know he is aware of the important ideas mentioned above even if he cannot give precise statements of them.

* * *

If you give the student an exercise such as:

$10 = \square + \hexagon$

then, since differently shaped frames are used, the replacement rule says that he <u>may</u> use different numbers in making the replacements. Now candidates for replacement are <u>pairs</u> of numbers rather than single numbers. You might speak of candidates for the box as <u>first numbers</u> and candidates for the hex as <u>second numbers</u>, since in left-to-right order the box appears first and the hexagon appears second. Thus, the pair of numbers with first number 7 and second number 3 satisfies the given expression. Students will soon recognize that for every pair which satisfies the expression (for example, 8 and 2), the pair obtained by interchanging first and second numbers (2 and 8) also satisfies the expression. This is an example of the commutative law for addition which says that the order does not matter in adding two

numbers. [Students do not need to learn the word "commutative", but they ought to recognize the universality of the rule.] Drawing a picture to keep track of which pairs of numbers work leads to coordinates and two dimensional graphs but is beyond the scope of this article.

* * *

Once students are accustomed to working with differently shaped frames and when they understand the replacement rule, you can use frames in many more ways. For example, when introducing students to the ideas of fractions, you can make precise the idea of, say, the number $\frac{1}{3}$ by telling them that this is the one number which satisfies the expression:

This is a step toward a mathematically sound formulation of the intuitive idea of cutting a pie into three pieces of the same size. We are not recommending that the pie idea be dropped but rather that it and other physical models be used to support the definition above.

The frame notation provides a convenient tool for the introduction of inverse operations. Students can learn that the number

5 - 3

is the number which satisfies the expression:

$5 = \bigcirc + 3$

Two ways of giving the same problem or, more precisely, two expressions which are satisfied by the same number are:

$5 - 3 = \diamond$ and $5 = \diamond + 3$

Similarly, when the student is introduced to division, he can learn that the number $6 \div 3$ is the number which satisfies the expression:

$6 = 3 \times \square$

When you present the idea of square root, the student can learn that $\sqrt{9}$ is a number which satisfies the expression:

$\bigcirc \times \bigcirc = 9$

You can move in another direction which is not common in grade school but which we think might be profitably undertaken, particularly in the later grades. You can give students problems which become more difficult as indicated by the following sequence. Of course, you would give many more problems than those listed, taking perhaps several weeks in going from problems like the first one to problems like the last one.

$5 + \square = 13$

$10 = \bigcirc + \bigcirc + 3$

$\frac{1}{2} = \left(2 \times \diamond\right) + \frac{1}{4}$

$\left(2 \times \square\right) + 6 = 12$

$\left(\frac{1}{3} \times \diamond\right) - 4 = 1$

$$\frac{\left(2 \times \square\right) + 3}{5} = 3$$

$\left(2 \times \square\right) + 1 = \square + 4$

You will recognize, of course, that students are solving equations in one variable. Let each student work out his own methods and discourage attempts to give procedures for obtaining answers, such as transposing or dividing

both sides, etc. A method which students work out for themselves will be more meaningful and longer remembered than one which they are told by a teacher or classmate. It should be noticed that here is another way of giving practice in computation as well as preparing a student for a real understanding of algebra. (In time a student may need to learn some formal algebraic procedures if he has not already discovered them. But there is no great urgency for this in the grades.)

In preparing lists of exercises in duplicated form (by spirit or stencil duplicating processes) you will find it convenient to use a template containing various sizes of frames (circular, triangular, square, and hexagonal). You can make such a template out of stiff cardboard or obtain one from a drafting supply house.

* * *

Frames play the same logical role as the letters of algebra. They are variables. However, we think they have a great psychological advantage over letters in introducing some ideas of algebra. When a child sees a letter x, he is too often tempted to ask, "What is x?" When he looks at a frame $\square$, he will almost automatically ask the correct question, "What can go into $\square$?" Many logical subtleties can be treated correctly but without mention through the use of frames.

In classroom work, you need a way or pronouncing the various shapes:

□, △, ▽, etc. We have used

"box"

"wedge"

"diamond"

etc.

In day to day use, these pronunciations are cumbersome and tiresome. If for variety you say "square" instead of "box", there is the risk of geometric and exponent associations which are undesirable. Recently we have hit on a solution. The value of frames is visual. It doesn't matter what you call them as long as it is short and as long as that sound does not already mean something else that will interfere. We propose calling them vocally by letter names as, for example, follows.

Write:	and say (phonetic spelling):
□	ekks
△	wie
▽	zee

This may seem bizarre at first. But after explaining to students along the lines above, it works surprisingly smoothly.

* * *

These suggestions do not prescribe a new mathematics curriculum for the elementary school. They do reflect the belief that children are willing to spend time on mathematics for the sheer intellectual challenge it can be made to offer. Games which are rich in mathematical (and not necessarily complex computational) content almost always contain challenges to which children respond.